Extended Constrained Controllability of Retarded Functional Differential Equations

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Abstract. In this work, Extended Constrained Controllability of Retarded Functional Differential Equations, we seek to establish some controllability conditions for Linear and Semi linear retarded delay systems, due to (Klamka, 2009). Necessary and sufficient conditions for local relative controllability of linear retarded delay system are established, using change of order of integration and a comparative approach of a system with and without delay. Sufficient conditions for local relative controllability of semi linear retarded delay system are also established, using the associated linear dynamical system. The result was obtained, using the generalized open mapping theorem. An example is provided to illustrate the theory.

KEY WORDS: controllability, linear, semi linear, constrained controls, retarded functional differential equations, Delay systems.

1 Introduction.

Controllability of dynamical systems is one of the fundamental concept in modern mathematical systems theory (Beata Sikora, 2003). Its theory is based on description of the dynamical (time-varying) and autonomous (time-invariant) systems (O. Sebakhy and M. M. Bayoumi, 1973). Roughly speaking, Controllability generally means the possibility of steering a dynamical system from an arbitrary initial state to another arbitrary final state, by using a set of admissible controls. Put differently, controllability is the property of being able to steer between two arbitrary points in the state space (Balachandran and Leelamani, 2006).

Volterra in 1928, formulated differential equations which took into account the past states of the system in his study of predator-prey models. Lately, Minossky incorporated "delay" in the equations he used to study ship movements. Delay dynamical systems can be encountered in many fields of science, and among other things, in industrial processes, medicine, biology and economy (Beata, 2003). Omar Sebakhy and M. M Bayoumi in 1973 studied Controllability of linear time-varying systems with delay in control. In 2003, Beata Sikora discussed Constrained controllability of dynamical systems with multiple delays in state. Klamka in 2004, formulated and proved sufficient conditions for constrained controllability of semi linear systems with point delay, and continued in that direction through 2009.

The aim of this work is to study the various constrained controllability conditions for the linear, semi linear control systems, and to significantly extend them to delay systems . Our approach is similar to that of Klamka (2008) for semi linear control system with delay in control, though, ours is an extension to retarded delay systems, with delay in both state and control.

2. System description

$$\dot{x} = A_0 x(t) + A_1 x(t-h) + B_0 u(t) + B_1 u(t-h) \tag{1.1}$$

$$\text{for } t \in [0,T], T > h$$

with zero initial conditions ;

$$x(t) = 0, u(t) = 0 \; for \; t \in [-h, 0] \tag{1.2}$$

and

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + F\big(x(t)\big) + B_0 u(t) + B_1 u(t-h) \tag{1.3}$$

$$for \; t \, \epsilon \, [0,T], T > h$$

with zero initial conditions ;

$$x(t) = 0, u(t) = 0 \; for \; t \in [-h, 0] \tag{1.4}$$

where ;

$x(t) \in \mathbb{R}^n$ is the instantaneous n −dimensional state vector.

$u(t) \in \mathbb{R}^m$ is the control function .

A_0, A_1 are the $(n \times n)$ −dimensional constant square matrix valued functions.

B_0, B_1 are $(n \times m)$ −dimensional constant column matrix valued functions.

F is n −vector function, that is continuous at zero (the origin)

H is the delay (time-lag)

The solution forms of (1.1) and (1.3), respectively are obtained, using variation of constant parameter method, thus;

$$x(t) = \int_0^t \phi(t,s)[A_1 x(s-h) + B_0 u(s) + B_1 u(s-h)]ds \tag{1.5}$$

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and

$$x(t) = \int_0^t \phi(t,s)\left[A_1 x(s-h) + F\big(x(s)\big) + B_0 u(s) + B_1 u(s-h)\right]ds \tag{1.6}$$

$where\ (t,s) = \varphi(t)\varphi^{-1}(s)$.

Let us define ;

$$Y(t,s) = \phi(t,s)\sum_{i=0}^{1} B_i \tag{1.7}$$

Therefore, the reachable set R, and the controllability grammian W are extracted from (1.5), thus ;

$$R(0,T) = \left\{\int_0^T Y(t,s)u(s)ds : u(t) \in R^m\right\} \tag{1.8}$$

and

$$W(0,T) = \int_0^T Y(t,s)Y^*(t,s)ds \tag{1.9}$$

To enable us focus our attention on the so-called relative controllability in the interval $[0,T]$, we shall first of all, introduce the notion of the attainable set at time $T > 0$ from zero initial conditions (1.2), denoted by $A_T(U_c)$ as in (Klamka, 1991).

$$A_T(U_c) = \{x \in X : x = x(T,u), u(t) \in U_c \text{for a.e } t \in [0,T]\} \tag{1.10}$$

where $x = x(T,u), t > 0$ is the unique solution of the delay system (1.3) with zero initial condition (1.4) and a given admissible control u.

and $*$ denotes transpose.

It is important to note that solutions of the systems (1.1) and (1.3), above exists under the assumptions made on the nonlinear term F(Klamka,1996,2006).

Now, using the concept in (1.9) to give the following definition (Klamka, 1991,1993, and H. Gorecki, A. Korytwski, J .E. Marshal, and K. Walton, 1992);

Definition 1(Controllability): The dynamical system (1.1) is said to be controllable on $[0,T]$ if for any initial function $x_0 \in C_n(0,T), and\ for\ every\ x_1 \in \mathbb{R}^n$, there exists an admissible control $u(t) \in \mathbb{R}^m$, which steers the response from x_0 at t_0 to any x_1 at T, (Balachandran, 1992).

Definition 2 (Complete state): The complete state of the system (1.1) at time t is given by $Z(t) = (x(t), x_t, u_t)$, where $x_t(s) = x(t+s), u_t(s) = u(t+s),\ s \in [-h, 0]$.

Definition 3 Dynamical system (1.1) is said to be controllable on J if it is proper on J (i.e,rank$[B_i, A_iB_i] = n$, $i = 0, 1, 2, 3, ..m$).

Definition 4 (Local Relative Controllability): Dynamical system (1.3) is said to be $U_c - locally\ relatively\ controllable\ on$ $[0,T]$ if the attainable set , say $A_T(U_c)$ contains a certain neighborhood of zero in the space X, Klamka (2007).

Definition 5(Global Relative Controllability): Dynamical system (1.3) is said to be $U_c - globally\ relatively\ controllable\ on$ $[0,T]$ if $A_T(U_c) = \mathbb{R}^n$ (Klamka, 2008).

3. Preliminaries.

Let n and m be positive integers, $\mathbb{R}$, the real line $(-\infty, +\infty)$. Let us define the space of real

n-tuples, with the inner product$\langle ., . \rangle$, let J be any interval on $\mathbb{R}$, then we denote the usual lebesque space of square summable functions from J to $\mathbb{R}^n$ $as\ l_\infty(J, \mathbb{R})$.

Let $\eta \geq h \geq 0$ be a given real number, let $C = C([-\eta, T], \mathbb{R}^n)$ be the space of continuous functions, and also bounded on $[-\eta, T]$, T is fixed.

If $x \in\ C([a, b], \mathbb{R}^n)$ for $a < b$, then for each fixed time $t \in [a, b]$, the symbol x_t denote an element of C, given by $x_t(s) = x(t+s), -h \leq s \leq 0$.

Similarly, for functions $u(t) \in l_\infty([a, b], \mathbb{R}^m)$, the symbol u_t denotes an element of l_∞ given by $u_t(s) = u(t+s), -h \leq s \leq 0$.

Note 1.

Controls of interest are;

(a) $U_{ad} = l_2([0,T], U_c)$
(b) $U = l_\infty([a,b], \mathbb{R}^m)$

Note 2.

$u(t)$ is assumed to be closed and convex, with vertex at zero, and with nonempty interior.

Below are some properties taken from the general theory of nonlinear operators in Banach spaces, and Kalman's rank condition used to establish the controllability of the linear system, and illustrate the results;

(a) The Generalized Open Mapping Theorem(In less general form useful for this purpose) : Let U *and* X be given Banach spaces, let Ω be an open subset of U, *containing* 0, *let* U_c be a closed and convex subset of U. Let $\beta: \Omega \to X$ be a nonlinear mapping , and suppose that on Ω, nonlinear mapping g has derivative $D\beta$, which is continuous at 0. Moreso, suppose $\beta(0) = 0$, and assume that the linear map $D\beta(0)$ *maps* U_C onto the whole space X, then there exist neighborhoods $N_0 \subset X$ about $0 \in X$, and $M_0 \subset \Omega$ about $0 \in U$ such that the non linear equation $x = \beta(u)$ has for each $x \in N_0$ at least, one solution $U \in M_0 \cap U_c$ where $M_0 \cap U_c$ is a so called conical neighborhood of zero in the space U.
(b) Kalman's rank condition : The linear dynamical system (1.1) is controllable if and only if rank$[B_i, A_i B_i, A_i^2 B_i, \ldots, A_i^{n-1} B_i] = n$,

where $A_i = [A_0, A_1]$, $B_i = [B_0, B_1]$, n = dimension of $A_i, i = 1, 2, \ldots m$.

4. Controllability condition 1

Here, we shall establish the necessary and sufficient conditions for constrained relative controllability of the linear dynamical system (1.1), and its equivalent system, without delay, given by the systems below;

$$\dot{x} = A_0 x(t) + A_1 x(t-h) + u(t) + B_1 u(t-h) \quad (1.11)$$

for $t \in [0, T]$, $T > h$

with zero initial conditions ;

$$x(t) = 0, u(t) = 0 \; for \; t \in [-h, 0] \quad (1.12)$$

and the equivalent system without delay ;

$$\dot{x}(x(t) = A_0 x(t) + A_1 x(t) + B_0 u(t) + B_1 u(t) \quad (1.13)$$

for $t \epsilon [0, T],\ T > h$

with zero initial conditions ;

$$x(t) = 0,\ u(t) = 0\ for\ t\ \in\ [-h, 0] \tag{1.14}$$

The solution form of the systems (1.11) and (1.13) are given by ;

$$x(t) = \int_0^t \emptyset(t,s)[A_1 x(s-h) + \ B_0\, u(s) + B_1 x(s-h\,)]ds \tag{1.15}$$

and ,

$$x(t) = \int_0^t \emptyset(t,s)[A_1 x(s) + \ B_0\, u(s) + B_1 x(s\,)]ds \tag{1.16}$$

$$where, \emptyset(t,s) = \varphi(t)\varphi^{-1}(s), \qquad (\varphi(t) = e^{A_i t},\ \varphi^{-1}(s) = e^{-A_i t}) \tag{1.17}$$

Lemma 1 : The linear dynamical system (1.11) is controllable if and only if $rank[B_i, A_i B_i, {A_i}^2 B_i, \ldots, {A_i}^{n-1} B_i] = n$, where A_i =[A_0 , A_1] , $B_i = [\ B_0\ , B_1\ ]$,

$n = dimension\ of\ A_i$.(i.e., properness condition (E.N Chukwu, 1979)

Proof

Recall that by definition (3), a system being proper on each interval J implies that the rank of the system is n (i.e, properness in E^n)(Chukwu, 1979).That is, system (1.11) is proper if and only if $C^T \varphi^{-1}(t) B_i(t)$ = 0 $\Longrightarrow a.e \Longrightarrow c\ =\ 0$ (1.18)

Assuming ,(1.18) is true.

Combining equation (1.18) and (1.17), we have ;

$C^T \varphi^{-1}(t) B_i(t)$ = 0 $\Longrightarrow a.e \Longrightarrow c\ =\ 0$ holds

Since if we Let y = $C^T \varphi^{-1}(t) B_i(t)$, we see that y is analytic (i.e, differentiable), then;

y^K= $C^T[(-A_i)^K e^{-A_i t} B_i]$, $K = Kth$ derivative of y. (1.19)

At $t = 0$, (1.19) becomes ;

$$C^T {A_i}^K B_i = 0\ , k\ =\ 0\ ,1\ ,2\ , \ldots . , n-1\ ; n\ is\ odd \Longrightarrow\ c\ =\ 0$$

But by orthogonality of A_i and B_j, we have;

$[B_{i\,,}A_iB_{j\,,}A_i{}^2B_{j\,,},..,A_i{}^{n-1}B_j]$ = 0 => $c \neq 0$, $i \neq j$

And by $C \in E^n$(since $c \neq 0$), we have that ;

$[B_{i\,,}A_iB_{i\,,}A_i{}^2B_{i\,,},..,A_i{}^{n-1}B_i]$ has rank n .

Conversely, let (on the contrary), $rank[B_{i\,,}A_iB_{i\,,}A_i{}^2B_{i\,,},..,A_i{}^{n-1}B_i] < n$, then there exists

$C \in E^n \ni c \neq 0$;

C^TB_i , $C^TA_iB_i$, $C^TA_i{}^2B_{i\,,}$, ..., $C^TA_i{}^{n-1}B_i$ = 0

Applying Cay lay Hamitian theorem, and by induction, we have ;

$$C^T e^{-A_it}B_i = C^T\sum_{i=0}^{\infty}\frac{(-A_i)^k}{k}t^kB_i = 0$$

This is a contradiction because, $C^T = 0$, $e^{-A_it} \neq 0$, $B \neq 0$, and by cancellation law, the result can not follow. Hence, $c = 0$ for the result to give the result of controllability ■

Result 1

Theorem 1 The linear dynamical system (1.11) is U_c $-$ *relatively controllable on* $[0,T]$ *for* $h < T$ *if and only if the linear dynamical system*(1.13) *is* V_c $-$ *controllable on* $[0,T]$,

where $V(t) \in V_{ad} = L_\infty([0,T],V_c)$ and $V_c = U_c \times U_c \times \ldots \times U_0 \in \mathbb{R}^{m(n+1)}$ is a given closed and convex cone, with nonempty interior, vertex at zero.

Proof

Firstly, controlling the delay in the solution, by changing the order of integration of (1.15), we have;

$$x(t,u) = \int_0^{t-h}\emptyset(t,s-h)A_1\,x(s)ds + \int_0^{t-h}\emptyset(t,s)B_0\,u(s)ds + \int_0^{t-h}\emptyset(t,s-h)B_1\,u(s)ds \tag{1.20}$$

Since the matrix $\emptyset(t,s-h)$ is always nonsingular, therefore do not change controllability property of dynamical systems, we can re-write (1.20) as;

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$$x(t,u) = \int_0^t \emptyset(t,s)A_1\, x(s)ds + \int_0^t \emptyset(t,s)B_0\, u(s)ds + \int_0^t \emptyset(t,s)B_1\, u(s)ds \qquad (1.21)$$

$where\ \emptyset(t,s)\ mob\ up\ h\ in\ (1.20).$

Hence, relative controllability of linear system (1.11) is actually equivalent to controllability of system (1.13) .Therefore, by lemma (1), the theorem follows ■

5 Controllability condition 2

Here, we shall show constrained local relative controllability on $[0,T]$ for the semi linear dynamical system (1.3);

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + F\big(x(t)\big) + B_0u(t) + B_1u(t-h) \qquad (1.22)$$

for $\in [0,T]$, $T < h$

with zero initial conditions ;

$$x(t) = 0, u(t) = 0, \text{ for } t \in [-h,T] \qquad (1.23)$$

The solution form of (1.22) is;

$$x(t,s) = \int_0^t \phi(t,s)\big[A_1x(s-h) + F\big(x(s)\big) + B_0u(s) + B_1u(s-h)\big]ds \qquad (1.24)$$

where $\emptyset(t,s) = \varphi(t)\varphi^{-1}(s)$,

Using the associated linear dynamical system, with single point delay in state and control.

Lemma 2 : Let $D_x x$ denote derivative of x with respect to u. Moreover, if $x(t,u)$ is continuousely differentiable with respect to it's argument, we have for each

$V \in L_\infty([0,T],U)$, $D_x x(t,u)(v) = Z(t)(t,u,v)$

Where the mapping t → $Z(t)(t,u,v)$ is the solution of the linear functional equation;

$$\dot{Z}(t) = A_0Z(t) + A_1Z(t-h) + D_x\left(F\big(x(t,u)\big)\right)Z(t) + B_0v(t) + B_1v(t-h) \qquad (1.25)$$

With zero initial conditions;

$$Z(0,u,v) = 0\ and\ v(t) = 0\ for\ t \in [-h,0)$$

Proof

Using equation (1.24), and the well known differentiability results;

$$x(t) = \int_0^t \phi(t,s)\big[A_1 x(s-h) + F\big(x(s)\big) + B_0 u(s) + B_1 u(s-h)\big]ds \tag{1.26}$$

Now, for a given admissible control u, (1.26) becomes;

$$x(t,u) = \int_0^t \phi(t,s)\big[A_1 x(s-h,u) + F\big(x(s,u)\big) + B_0 u(s) + B_1 u(s-h,)\big]ds \tag{1.27}$$

Taking derivative of (1.27), with respect to u, we have ;

$$D_u x(t,u) = \int_0^t \emptyset(t,s) A_1 D_u x(s-h,u)ds + \int_0^t \emptyset(t,s) D_x\, F\big(x(s,u)\big).\, D_u x(s,u)\, ds$$

$$+ \int_0^t \emptyset(t,s) B_0 u(s) ds + \int_0^t \phi(t,s) B_1 u(s-h) ds \tag{1.28}$$

Then, taking derivative of (1.28) with respect to to t, we have ;

$$\frac{d}{dt}[D_u x(t,u)(v)] = A_1 D_u x(t-h,u) + D_x F\big(x(s,u)\big).\, D_u x(s,u) + B_0 u(t) + B_1 u(t-h)ds$$

$$+ \int_0^t \frac{d}{dt}\emptyset(t,s) B_0 u(s) dsv + \int_0^t \frac{d}{dt}\phi(t,s) B_1 u(s-h) ds\, v$$

$$+ \int_0^t \frac{d}{dt}\emptyset(t,s) D_x F\big(x(s,u)\big).\, D_u x(s,u) dsv$$

$$+\int_0^t \frac{d}{dt}\emptyset(t,s)D_u x(s-h,u)dsv$$

(liebniz rule)

Since by assumption , $\emptyset(\boldsymbol{t},\boldsymbol{s})$ is differentiable whenever this semigroup $\varphi(t)$ is differentiable, then ;

$\frac{d}{dt}\emptyset(t,s) = A_0\emptyset(t,s)$, and we have ;

$$\frac{d}{dt}[D_u x(t,u)(v)] = A_1 D_u x(t-h,u)v + D_x F\big(x(t,u)\big).D_u x(t,u)v$$

$$+\ B_0 u(t) + B_1 u(t-h)$$

$$+\int_0^t A_0\emptyset(t,s)B_0 u(s)dsv + \int_0^t A_0\phi(t,s)B_1 u(s-h)ds\,v$$

$$+\int_0^t A_0\,\emptyset(t,s)D_x F\big(x(s,u)\big).D_u x(s,u)dsv$$

$$+\int_0^t A_0\,\emptyset(t,s)A_1 D_u x(s-h,u)dsv$$

(1.29)

Now, from the (lemma 2) , we have that ;

$$D_u x(t;u)(v) = Z(t,u,v), then\ \frac{d}{dt}[D_u x(t,u)(v)] = \dot{Z}(t,u,v)$$

Therefore, factorizing and comparing (1.29) with (1.25), we have ;

$\dot{Z}(t) = A_0 Z(t) + A_1 Z(t-h) + D_x\left(F\big(x(t,u)\big)\right)Z(t) + B_0 v(t) + B_1 v(t-h)$

Hence, lemma 2 follows ■

Therefore, the associated linear dynamical system with single point delay in state and control is

$\dot{Z}(t) = CZ(t) + A_1 Z(t-h) + B_0 V(t) + B_1 V(t-h)$ (1.30)

$$for\ t \in [0, T], T\ < h$$

With zero initial conditions ;

$$Z(t) = 0\ , V(t) = 0\ , for\ t\ \in [-h, 0]$$

Where $where\ C = A_0 + D_x F(0)$ (1.31)

Result 2

Theorem 2 Suppose that;

(i) $F(0) = 0$
(ii) $U_c \subset U$, is a closed and convex cone with vertex at zero
(iii) The associated linear dynamical system (1.30), with point delay in state and control is $U_c - globally\ controllable\ on\ [0, T]$.Then , the semi linear dynamical system (1.22) is $U_c - locally\ relatively\ controllable\ on\ [0, T]$

Proof

Firstly;

Let $\beta: L_\infty([0, T], C) \rightarrow X$ be the nonlinear map for the system in equation (1.25), whose continuous derivative is the linear map F, defined thus ; $Fv = Z(t)(T, v)$

Secondly;

We show that;

(a) The nonlinear map β transforms conical neighborhood of zero in the set of admissible Controls U_{ad} onto some neighborhood of zero in the space, X, and,
(b) That the semi linear system is $U_c - locally\ relatively\ controllable\ in\ [0, T]$.

To show (a), it suffices to show that β satisfies all the assumptions of the generalized open mapping thorem (a)

Observe that, by assumption (iii), the linear map F is clearly surjective that is, it maps the cone, U_c onto the whole space$X$, that is $A_T(U_c) = X$, (satisfied by definition 5), and by lemma 2, $D_\beta(0) = H$, which are the assumptions of the generalized open mapping thorem(a), therefore, β has satisfied (a), above.

Again, by definition 4, (b), above is satisfied.

Having established (a) and (b), our theorem follows immediately ■

6. Application.

This section contains numerical example illustrating the theoretical analysis.

Example 1, 2 illustrates the linear, semi linear systems demonstrated in this work, respectively.

Example 1: Let us consider the dynamical system below, with delay h = 1

$$\dot{x_1}(t) = x_1(t) + x(t)$$

$$\dot{x_2}(t) = -2x_2(t-1) + u(t-1) \tag{1.32}$$

We see clearly, from the above system that;

$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$, $B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Required to show; (1) is controllable.

Solution.

Suffices to show that rank $[B_i, A_i B_i] = n$,

Where $A_i = \{A_0, A_1\}$,$B_i = \{B_0, B_1\}$.

Assume that $U_c = \mathbb{R}^+$, and the set of admissible controls $U_{ad} = ([0,T], \mathbb{R}^+)$

Now,

rank $[B_i, A_i B_i]$

=rank $[B_0, B_1, (A_0, A_1)(B_0, B_1)]$

=rank$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\right)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]$

=rank$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right]$

= rank$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$ = 2 $= n$, where $n = 2 = dim A_i$.

Hence, example (1) is controllable.

Example 2

$\dot{x_1}(t) = x_1(t) + u(t)$

$\dot{x_2}(t) = -x_2(t-1) + u(t-1) + sinx_1(t)$ (1.33)

We see clearly also, from the above (1.33) that;

A_0, A_1, B_0, B_1 are as in example (1) .

$F(\tilde{x}) = F(x_1, x_2) = \begin{pmatrix} 0 \\ sinx_1(t) \end{pmatrix}$, where $\tilde{x} = (x_1, x_2), a\ vector$.

Assume also that $U_c = \mathbb{R}^+$, and the set of admissible controls $U_{ad} = L_\infty([O,T], \mathbb{R}^+)$.

Now,

$$F(0) = F(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$D_x F(\tilde{x}) = \begin{pmatrix} 0 & 0 \\ cosx_1 & 0 \end{pmatrix}$, where $D_x F(\tilde{x}) = D_{(x_1,x_2)} F(x_1, x_2)$

$$D_{\tilde{x}} F(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

This implies that;

C = $A_0 + D_{\tilde{x}} F(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Therefore, rank $[B_i, A_i B_i]$

= rank $[B_0, B_1, (C, A_1)(B_0, B_1)]$, where $A_1 = \{C, A_1\}$

= rank $[B_0, B_1, CB_0, A_1 B_1]$

= rank$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

= rank$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

= rank$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$=2 $= n$, where $n = 2 = dimA_i$.

Hence, example 2 is controllable.

Proven that the linear and semi linear systems in example (1) and (2) are controllable.

7 .Conclusion

In this work, sufficient conditions for constrained local relative controllability, near the origin for linear and semi linear finite dimensional delay system, with single time-variable point delay in state and control have been established.

For the linear system, equivalence of a system, with and without delay was established for local relative controllability.

For the semi linear system, the associated linear dynamical system was used to establish local relative controllability.

Examples are used to illustrate the theoretical analysis, by using an already existing computable criterion, known as Kalman's Rank Condition.

8.Furtherwork

Since this method worked for Control System (developed and verified by Klamka), I extended it to Retarded Delay System, I recommend that it should also be extended to Neutral System (if possible).

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